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Motion of a triple junction

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The motion of a triple junction is investigated. We consider only the two-dimensional case and assume that initially the fluid is in the form of three inviscid wedges. A similarity solution is determined which accounts for a balance of force at the triple junction. This similarity solution is computed numerically using boundary integral methods. Results are presented for different initial wedge angles and surface tension ratios. In particular the location of the triple junction and the resulting capillary waves along the interfaces are discussed.

1. Introduction

Many interesting and important physical processes involve the spreading of one liquid over another. There are applications to many fields. Specific examples are liquid waste spills on bodies of water (e.g. oil spreading on the sea or chemical waste spills in ponds) and aerosol delivery of bronchial medicated mists. Although one can think of several relevant but different situations to study, it is clear that the relationship between the surface tensions of the liquid interfaces is one of the dominant factors in determining how the liquids spread.

Consider what occurs when a liquid drop is placed in contact with the interface of a second liquid. After the initial instant of contact, a three-phase (gas/liquid/liquid) point occurs in two dimensions (this is a three-phase line in three dimensions) if the liquids are immiscible. We will refer to this three-phase point as the triple junction. The surface tension forces at this triple junction influence how the drop spreads along the interface of the base liquid. In particular, for low viscosity fluids, the initial motion might be expected to be governed primarily by inertial and surface tension forces. In addition, near the triple junction an initial distribution of the fluids into three wedges may be a reasonable local approximation. This initial geometry is determined by the drop placement process but, after the triple junction is formed, the local interface shape will adjust so as to properly balance all local forces (see the review article Eggers (1997) for the related problem of the rupture of a fluid thread). Now consider a situation where surfactant is deposited onto the interface of a droplet resting in equilibrium on the interface of a second liquid. As in the previous problem, the local geometry at the triple junction (in two dimensions) can be described by the intersection of three fluid wedges. If the adsorption rate of surfactant onto the interface is large (e.g. there is no diffusion transport limitation), the surface tension will change more rapidly than the deformation of the interfaces. The result will be an imbalance of force at the triple junction resulting in the motion of the interfaces.

Again, an inviscid solution might be expected to describe the early dynamics of the interfaces, if the surface forces dominate the viscous forces (see Daniel, Chaudhury & Chen (2001) for the related problem of the motion of droplets by phase change on a gradient surface). In the two problems described above, the local dynamics appears to be related to the motion of three fluid wedges driven by a relation between the forces at the triple junction. Although the full dynamics of the interfaces is much more complicated than implied here, a study of this local canonical triple-junction problem may give some insight into the spreading process.

A measure of the effect of the different surface tension forces is given by the spreading coefficient. For a droplet of liquid in air placed on top of a base liquid, positive spreading coefficient, S, means that the surface tension of the base liquid with air, σ_{la}, is larger than the sum of the surface tension of the droplet with air, σ_{da}, and the surface tension between the liquid and the droplet, σ_{ld}, i.e. $S = \sigma_{la} - \sigma_{la} - \sigma_{da} > 0$. Hence the droplet will completely wet the second liquid, i.e. without any additional assumptions it will completely cover the interface of the base liquid. Such a situation has been investigated by DiPietro, Huh & Cox (1978), DiPietro & Cox (1980) and Foda & Cox (1980) who developed a theory for the spreading of a droplet in the completely wetting case. Their model included the additional effect of a leading precursor (a monolayer) film. The addition of the precursor film to the model allowed them to obtain steady and similarity solutions. The positive spreading coefficient case was also investigated analytically by Joanny (1987) and Brochard-Wyart, Debregeas & de Gennes (1996) and experimentally by Fraaije & Cazabat (1989).

Less work has been done for negative spreading coefficients, $S < 0$. In this case an equilibrium situation is possible without the additional assumption of a monolayer precursor film. An example of such a situation is a water droplet on top of a pool of carbon tetrachloride (or almost any organic liquid). Equilibrium solutions of droplets resting on a liquid interface were computed by Pujado & Scriven (1972). Recently a lubrication model was used by Wilson & Williams (1997) to study the two-liquid coating problem in the negative spreading coefficient case when a zero net force assumption is made at the triple junction of the three liquid interfaces for all time. Also recently, Kriegsmann (1999) studied the motion of a droplet as it flows on top of a liquid film on an inclined plane with a similar assumption.

Here the canonical situation discussed above where the liquids are initially in the form of three wedges is considered. The zero net force condition is not initially satisfied at the triple junction but it will be satisfied for time positive. This then results in the motion of the interfaces and the triple junction. It will be shown that there is a similarity solution of the problem and this solution will be determined numerically. The solutions obtained are all time dependent. The zero net force conditions at the triple junction can be satisfied for a range of values of the surface tensions since the interfaces are allowed to deform. Hence the definition of a spreading coefficient loses some of its meaning here since strictly speaking there is no coating liquid spreading over a base liquid. It is therefore more appropriate to ask when the balance of force can be satisfied at the contact point. The answer is clearly that the balance of force can always be satisfied as long as the sum of any two of the surface tensions is larger than the third surface tension.

The basic scaling and the solution technique used here parallel the work of Keller & Miksis (1983) who studied the motion of a single inviscid wedge of fluid. Unlike the problem considered here, the initial vertex of their wedge was confined to remain along one of the coordinate axis, whereas here the location of the triple junction in the plane is part of the solution of the problem. In another related work Keller, Milewski

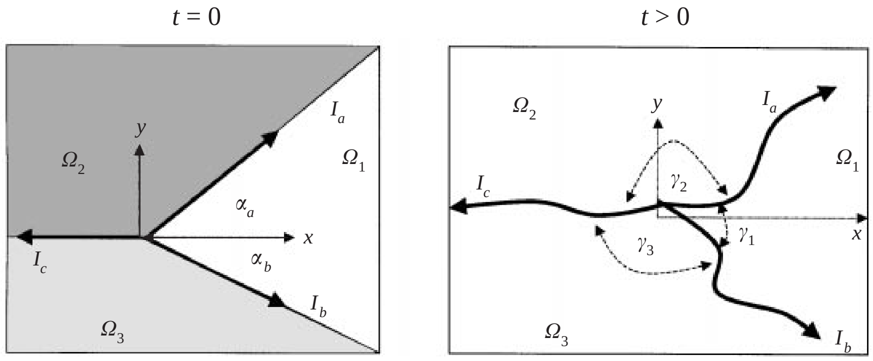

FIGURE 1. Initial configuration at $t = 0$ and a dynamic triple junction for $t > 0$.

& Vanden-Broeck (2000) considered the corresponding merging of two fluid wedges using a similarity solution. Also note that the present problem can be extended to Stokes flows by following the formulation in Miksis & Vanden-Broeck (1999).

2. Formulation

Suppose that initially, three inviscid liquid wedges meet at a common point, the triple junction. We wish to study the dynamics of these wedges given that for positive time, the interfaces must satisfy the boundary conditions of zero net force at the triple junction. Although this is an equilibrium condition, we will assume that it holds for all time.

Initially the three different fluids are in the form of three wedges, see figure 1. Denote these fluid regions by Ω_1, Ω_2 and Ω_3. Let the three interfaces be denoted by I_a, I_b, and I_c. Suppose that I_a and I_b bounds Ω_1, that I_a and I_c bound Ω_2 and that I_b and I_c bound Ω_3. We will assume potential flow in each region. Hence letting Φ^i, $i = 1, 2, 3$, be the velocity potential in each of the regions Ω_i, we have

$$\nabla^2 \Phi^i = 0 \tag{2.1}$$

for $(x, y) \in \Omega_i$.

Each interface I_j is assumed to have a surface tension σ_j, where $j = a, b, c$, and each fluid region Ω_i is assumed to have a density ρ_i where $i = 1, 2, 3$. Hence using Bernoulli's theorem and the fact that the jump in normal stress across the interface is equal to the mean curvature times the surface tension, we find that along interface I_a which separates Ω_1 and Ω_2,

$$\rho_2 \frac{\partial \Phi^2}{\partial t} - \rho_1 \frac{\partial \Phi^1}{\partial t} + \frac{\rho_2}{2}(\nabla \Phi^2)^2 - \frac{\rho_1}{2}(\nabla \Phi^1)^2 = \sigma_a \kappa_a, \tag{2.2}$$

where κ_a is the mean curvature of interface I_a. Equations similar to (2.2) can also be derived along I_b and I_c. In addition we also have the kinematic boundary condition (i.e. the normal velocity of the fluid equals the normal velocity of the boundary) along each interface. In order to complete the formulation of the problem we still need to specify boundary conditions at the triple junction and at infinity.

The boundary conditions at the triple junction for $t > 0$ are given by a balance of surface tension forces, i.e. forcing zero net force. Suppose that we let γ_i, $i = 1, 2, 3$, represent the angles between the interfaces at the triple junction in region Ω_i, see figure 1. Then the balance of force conditions at the triple junction relate the angles γ_i

to the surface tension by the Neumann triangle, e.g. see Rowlinson & Widom (1989). The Neumann triangle is simply a triangle with sides of lengths σ_j, $j = a, b, c$, and internal angles $\pi - \gamma_i$, $i = 1, 2, 3$, where these angles are bounded by the same sides as γ_i is in figure 1. As noted in the previous section, the Neumann triangle can only exist (i.e. the forces can only balance) if the sum of the lengths of any two sides (σ_j) is larger than the third. Hence, using the law of cosines on the Neumann triangle, the zero net force condition implies

$$\cos(\gamma_1) = \frac{\sigma_c^2 - \sigma_b^2 - \sigma_a^2}{2\sigma_a\sigma_b}, \quad \cos(\gamma_2) = \frac{\sigma_b^2 - \sigma_c^2 - \sigma_a^2}{2\sigma_a\sigma_c}, \quad \cos(\gamma_3) = \frac{\sigma_a^2 - \sigma_b^2 - \sigma_c^2}{2\sigma_c\sigma_b}. \tag{2.3}$$

Note that although there are three conditions in (2.3), one is dependent on the other two since the angles at the triple junction must add up to 2π, i.e. $\gamma_1 + \gamma_2 + \gamma_3 = 2\pi$. These conditions are assumed to hold for $t > 0$. They force the interface to change from the initial data of a straight sided wedge to something evolving with non-straight sides. To be specific about the initial data, we assume that initially I_a makes an angle α_a with the x-axis, that I_b makes an angle α_b with the x-axis and that I_c lies along the negative x-axis, see figure 1. Note that this implies that the initial triple junction is at the origin and that these angles will be preserved far from the origin for all time (i.e. these angles define boundary conditions at infinity).

We note that there is no natural length scale, therefore we can look for a similarity solution. Following Keller & Miksis (1983), we introduce the similarity variables

$$\xi = x(\rho_1/\sigma_a t^2)^{1/3}, \quad \eta = y(\rho_1/\sigma_a t^2)^{1/3}, \quad \Phi^i = (\sigma_a^2 t/\rho_1^2)^{1/3}\phi^i. \tag{2.4}$$

Note that we scale all variables using σ_a and ρ_1. Hence we need to introduce the density ratios $\beta_i = \rho_i/\rho_1$, $i = 2, 3$, and the surface tension ratios $\Sigma_j = \sigma_j/\sigma_a$, where $j = b, c$.

After applying (2.4) to the equations of motion and boundary conditions, we find that the potential ϕ^i still satisfies Laplace's equation in each region Ω^i (now in the (ξ, η) similarity plane). In addition equation (2.2) becomes

$$\beta_2\left\{\tfrac{1}{3}\phi^2 - \tfrac{2}{3}(\xi\phi^2_\xi + \eta\phi^2_\eta)\right\} - \left\{\tfrac{1}{3}\phi^1 - \tfrac{2}{3}(\xi\phi^1_\xi + \eta\phi^1_\eta)\right\} + \tfrac{1}{2}\left\{\beta_2(\nabla\phi^2)^2 - (\nabla\phi^1)^2\right\} = \kappa_a. \tag{2.5}$$

Here κ_a represents as before the mean curvature of I_a but now in the similarity plane. We will still use Ω_i, $i = 1, 2, 3$, to refer to each of the fluid regions and I_j, $j = a, b, c$, to refer to each of the interfaces in the similarity plane.

Note that along each interface there are four unknowns giving a total of twelve unknown functions. These are the parameterization of each interface, $\xi^j(s)$, $\eta^j(s)$, $j = a, b, c$, where s is the arclength, and the six potential values, one on each side of each interface. For example, along interface I_a there are unknown potentials ϕ_a^1 and ϕ_a^2 on each side of the interface. There is no reason to assume that these potentials are the same across the interface since, in potential flow, the tangential velocity on each side of the interface does not have to be continuous. The only relation we assume between each of these potential functions is that initially they are all zero, i.e. no flow, so they are zero at infinity in each region for all time. Hence this is the boundary condition to be used at $|s| = \infty$ along each interface. We define the arclength such that $s = 0$ at the triple junction and assume that the arclength increases to positive infinity along interfaces I_a and I_c but decreases to minus infinity along interface I_b. This is just a minor technical point necessary for the parameterization.

Our plan is to solve for the twelve unknown functions along the three interfaces by reformulating the problem as a system of integro-differential equations. Note that there are two sets of integral equations for each of the potentials on each surface (six equations), the arclength condition on each interface (three equations) plus the balance of normal force as in equation (2.5) (three equations). Hence the number of equations is equal to the number of unknowns. Here the arclength condition means that the square of the derivatives of η and ξ with respect to the arclength adds up to one along each interface.

As an example of the resulting integral equations, suppose we consider the form of the equation on the Ω_1 side of interface I_a. In this case we find that

$$\tfrac{1}{2}\phi_a^1(s) = \int_0^\infty \left\{ \phi_a^1(\hat{s}) \frac{\partial g}{\partial n}(\xi^a(\hat{s}), \eta^a(\hat{s}); \xi^a(s), \eta^a(s)) - g(\xi^a(\hat{s}), \eta^a(\hat{s}); \xi^a(s), \eta^a(s)) \frac{\partial \phi_a^1}{\partial n}(\hat{s}) \right\} \mathrm{d}\hat{s}$$
$$+ \int_\infty^0 \left\{ \phi_b^1(\hat{s}) \frac{\partial g}{\partial n}(\xi^b(\hat{s}), \eta^b(\hat{s}); \xi^a(s), \eta^a(s)) - g(\xi^b(\hat{s}), \eta^b(\hat{s}); \xi^a(s), \eta^a(s)) \frac{\partial \phi_b^1}{\partial n}(\hat{s}) \right\} \mathrm{d}\hat{s}, \tag{2.6}$$

where the free-space Green's function g is given by

$$g(\xi, \eta; \hat{\xi}, \hat{\eta}) = \frac{1}{2\pi} \log [(\xi - \hat{\xi})^2 + (\eta - \hat{\eta})^2]^{1/2}. \tag{2.7}$$

In equation (2.6), partial derivatives with respect to n are in the direction of the unit normal out of region Ω_1 and $s > 0$. The normal derivatives of the potential ϕ are given by the kinematic condition,

$$\frac{\partial \phi_j^1}{\partial n} = \frac{2}{3}\left(\eta^j \frac{\mathrm{d}\xi^j}{\mathrm{d}s} - \xi^j \frac{\mathrm{d}\eta^j}{\mathrm{d}s} \right), \tag{2.8}$$

where $j = a$ or b. Notice that equation (2.6) is singular along I_a. In a like manner, we can obtain five additional integral equations similar to (2.6), one on each side of the interface. We parallel the numerical method presented in Keller & Miksis (1983) for the discretization of (2.5) and (2.6) (see also Keller *et al.* 2000).

3. Boundary conditions

Here we discuss in more detail the boundary conditions we will use for the numerical solution of this problem. Note that since there are twelve unknowns in this problem, we need twelve conditions at the triple junction plus four conditions on each interface as $|s| \to \infty$. Given these conditions plus the integral equations and differential equations similar to (2.5) and (2.6) along each interface, we will discretize the problem and determine all the unknowns by solving the resulting nonlinear algebraic equations numerically.

First consider the limit as $|s| \to \infty$ along each of the interfaces. Because of the initial data we set the potentials, ϕ_a^1, ϕ_b^1, ϕ_a^2, ϕ_c^2, ϕ_b^3, ϕ_c^3, all to zero (six conditions) as $|s| \to \infty$ and we force the interface to lie on a line (three conditions) at infinity, i.e. on the interface I_j we set at infinity

$$\eta^j \cos(\alpha_j) - \xi^j \sin(\alpha_j) = 0, \tag{3.1}$$

where α_j, $j = a, b$, is the initial wedge angle that interface I_j makes with the ξ-axis. For interface I_c we set $\eta^c = 0$ at $s = \infty$. The final three conditions at infinity are given by forcing the arclength condition at the last point with one-sided derivatives.

At the contact point we also need twelve conditions. Four of these are given by forcing the interfaces to be in contact at this point, i.e. $\xi^a(0) = \xi^b(0) = \xi^c(0)$ and $\eta^a(0) = \eta^b(0) = \eta^c(0)$ (here the superscripts refer to that variable on a specific interface). Two additional conditions are given by the two Neumann conditions (as in equation (2.3)). This leaves six conditions yet to be specified.

The final six conditions are found by forcing the velocity vector along the two bounding interfaces of Ω_i to be continuous at the triple junction. That is, we need the velocity of the triple junction to be the same no matter how it is approached. This implies that we need both the ξ- and η-components of the velocity vector along the interface and at the triple junction to be continuous if we stay on the Ω_i side of the interfaces. Note that only the continuity of $\nabla\phi$ needs to be considered here since the velocity vector is proportional to it in the similarity plane. Now write the ξ- and η-components of $\nabla\phi$ on an interface in terms of the tangential $(\partial\phi/\partial s)$ and normal $(\partial\phi/\partial n)$ derivatives of ϕ (drop the superscripts and subscripts for the moment):

$$\begin{aligned}\nabla\phi &= \boldsymbol{n}\frac{\partial\phi}{\partial n} + \boldsymbol{t}\frac{\partial\phi}{\partial s} \\ &= \left(-\frac{\mathrm{d}\eta}{\mathrm{d}s}, \frac{\mathrm{d}\xi}{\mathrm{d}s}\right)\frac{\partial\phi}{\partial n} + \left(\frac{\mathrm{d}\xi}{\mathrm{d}s}, \frac{\mathrm{d}\eta}{\mathrm{d}s}\right)\frac{\partial\phi}{\partial s} \\ &= \left(-\frac{\mathrm{d}\eta}{\mathrm{d}s}\frac{\partial\phi}{\partial n} + \frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\partial\phi}{\partial s}, \frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\partial\phi}{\partial n} + \frac{\mathrm{d}\eta}{\mathrm{d}s}\frac{\partial\phi}{\partial s}\right). \end{aligned} \tag{3.2}$$

Here $\boldsymbol{n}$ and $\boldsymbol{t}$ are the unit normal and tangent to the interface and the above notation refers to the ξ- and η-components of velocity. The aim is to evaluate (3.2) at the triple junction $s = 0$.

From the kinematic condition in similarity form, we also have the condition that

$$\frac{\partial\phi}{\partial n} = \frac{2}{3}\left(-\xi\frac{\mathrm{d}\eta}{\mathrm{d}s} + \eta\frac{\mathrm{d}\xi}{\mathrm{d}s}\right). \tag{3.3}$$

This relation can be used in equation (3.2) in place of the normal derivative of ϕ.

Now each component of the vector in equation (3.2) must be continuous at the triple junction as it is approached along either of the interfaces bounding a given region. For Ω_1, this implies that the limits of $\nabla\phi$, given by (3.2), are equal as s approaches the triple junction ($s = 0$) along I_a and I_b. Hence the jump in the ξ-component of velocity is continuous if

$$\left[-\frac{\mathrm{d}\eta}{\mathrm{d}s}\frac{2}{3}\left(-\xi\frac{\mathrm{d}\eta}{\mathrm{d}s} + \eta\frac{\mathrm{d}\xi}{\mathrm{d}s}\right) + \frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{\partial\phi^1}{\partial s}\right]_b^a = 0 \tag{3.4}$$

at $s = 0$. And for the η-component of velocity we need

$$\left[\frac{\mathrm{d}\xi}{\mathrm{d}s}\frac{2}{3}\left(-\xi\frac{\mathrm{d}\eta}{\mathrm{d}s} + \eta\frac{\mathrm{d}\xi}{\mathrm{d}s}\right) + \frac{\mathrm{d}\eta}{\mathrm{d}s}\frac{\partial\phi^1}{\partial s}\right]_b^a = 0. \tag{3.5}$$

Here the bracket notation is introduced to represent the jump from interface I_a to interface I_b at the contact point. Note that (3.4) and (3.5) are evaluated on the Ω_1 side of each of the interfaces.

Now at the triple junction the arclength condition must hold,

$$\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 = 1. \tag{3.6}$$

Using this in equation (3.4) along with the fact that there is contact at $s = 0$ gives,

$$\left[\left\{-\frac{2}{3}\left(\xi\frac{d\xi}{ds} + \eta\frac{d\eta}{ds}\right) + \frac{\partial\phi^1}{\partial s}\right\}\frac{d\xi}{ds}\right]_b^a = 0, \tag{3.7}$$

while equation (3.5) gives

$$\left[\left\{-\frac{2}{3}\left(\xi\frac{d\xi}{ds} + \eta\frac{d\eta}{ds}\right) + \frac{\partial\phi^1}{\partial s}\right\}\frac{d\eta}{ds}\right]_b^a = 0. \tag{3.8}$$

Note that the system given in (3.7) and (3.8) can be rewritten as (with obvious meaning)

$$A\frac{d\xi^a}{ds} = B\frac{d\xi^b}{ds}, \quad A\frac{d\eta^a}{ds} = B\frac{d\eta^b}{ds}. \tag{3.9}$$

Suppose we assume that the coefficients

$$A = -\frac{2}{3}\left(\xi^a\frac{d\xi^a}{ds} + \eta^a\frac{d\eta^a}{ds}\right) + \frac{\partial\phi_a^1}{\partial s},$$

evaluated at $s = 0$ along interface I_a, and

$$B = -\frac{2}{3}\left(\xi^b\frac{d\xi^b}{ds} + \eta^b\frac{d\eta^b}{ds}\right) + \frac{\partial\phi_b^1}{\partial s},$$

evaluated at $s = 0$ along the interface I_b, are both not zero. Now multiply the first equations (3.9) by $d\eta^a/ds$ and the second of equations (3.9) by $d\xi^a/ds$ and subtract the result. We find that for $B \neq 0$,

$$\frac{d\xi^b}{ds}\frac{d\eta^a}{ds} = \frac{d\xi^a}{ds}\frac{d\eta^b}{ds}. \tag{3.10}$$

Equation (3.10) states that at the triple junction, the normal from interface I_b is perpendicular to the tangent from interface I_a. This can only happen if the interface is smooth at the contact point, implying that a general triple junction is impossible. Hence, the condition that $A = B = 0$ is required at a triple junction.

Therefore we must have on interface I_a and I_b at $s = 0$

$$\frac{\partial\phi_j^1}{\partial s} = \frac{2}{3}\left(\xi^j\frac{d\xi^j}{ds} + \eta^j\frac{d\eta^j}{ds}\right), \tag{3.11}$$

$j = a, b$. Note that within Ω_1 the tangential derivative does not have to be continuous at $s = 0$. But since the result only depends on which interface is being considered, a similar derivation for the other regions implies that the tangential velocities on each side of the same interface are equal at the triple junction. Hence conditions similar to (3.11) hold along each interface, giving a total of six additional boundary conditions at the triple junction.

All the necessary boundary conditions to solve for the interfaces plus the potentials along them have now been determined. A solution of the resulting integro-differential system can be obtained by discretizing the system as outlined in Keller & Miksis (1983) for the case of a single wedge. As noted there, the convergence of the numerical

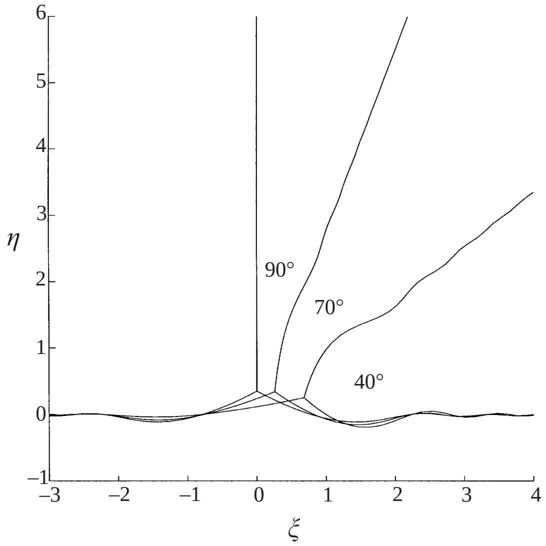

FIGURE 2. Interface profiles for wedge angle $\alpha_a = 90°, 70°, 40°$. Here we set $\alpha_b = 0°$, $\Sigma_b = \Sigma_c = 1$, $\beta_2 = \beta_3 = 1$.

method is very sensitive both to how the infinite integrals in the integral equation are truncated and to the fact that surface waves appear on the interfaces with a wavelength that decreases with the distance along the interface squared and an amplitude that decreases with distance to the 7/2 power. A parallel analysis gives the same small-amplitude decay rate and wavelength decrease with distance from the triple junction. Hence many mesh points along the interface are needed to resolve these decaying waves.

4. Results

To begin our numerical study of similarity solutions associated with this triple junction problem, we set the density and surface tension ratios equal to one, i.e. $\beta_2 = \beta_3 = \Sigma_b = \Sigma_c = 1$, and $\alpha_b = 0$, and solve the free boundary problem for various values of α_a. Note that by equating all of the surface tensions, equation (2.3) implies that $\gamma_1 = \gamma_2 = \gamma_3 = 2\pi/3$. What is not known is the location in the (ξ, η)-plane of the triple junction plus its orientation. These are found by solving the free boundary problem. The results, illustrated in figure 2, show that for $\alpha_a = 90°$ the solution is symmetric about the ξ-axis. No waves are observed along interface I_a but they are obvious along I_b and I_c.

As the angle α_a decreases from 90°, we see that waves form along interface I_a, with their amplitude increasing as α_a decreases. Let (ξ_0, η_0) denote the location of the triple junction. Then as α_a decreases, ξ_0 increases but η_0 has a slight decrease. Note that in the physical (x, y)-plane, the triple junction is moving with a velocity proportional to $t^{-1/3}$ (see equation (2.4)), in the direction of the vector $\xi_0 \boldsymbol{i} + \eta_0 \boldsymbol{j}$. Hence during its initial reorientation, the triple junction has an infinite velocity but then the speed decreases with time. Also note that with increasing distance from the triple junction, the wavelength of the interfacial waves decreases. This is consistent with the small-amplitude analysis noted in the previous section.

Now set $\alpha_a = 50°$ and keep the other parameters the same as in figure 2. Suppose

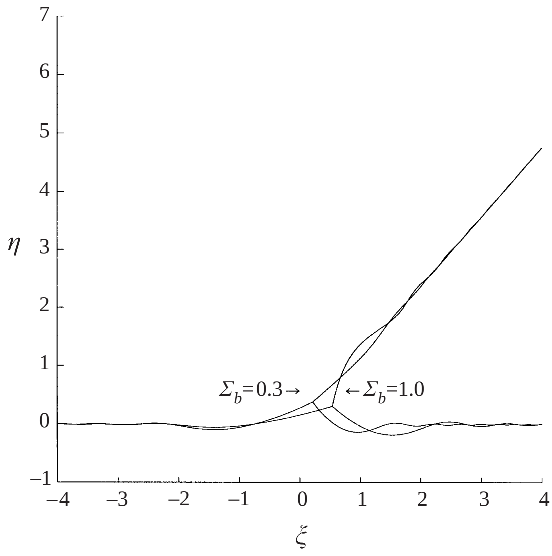

FIGURE 3. Wedge profiles for different surface tensions along interface I_b. Here we set $\alpha_a = 50°$, $\alpha_b = 0°$, $\Sigma_c = 1$, $\beta_2 = \beta_3 = 1$.

now that the surface tension ratio is decreased from $\Sigma_b = 1.0$ to $\Sigma_b = 0.3$ while keeping the other parameters constant. This situation is illustrated in figure 3. As Σ_b decreases, observe the change in the amplitude of the waves along interface I_a and I_c. In particular, it appears that the amplitude of the waves is less along the interface I_a for smaller Σ_b. This is probably because the change in the location of the triple junction from the origin, i.e. its initial position in the physical (x, y)-plane, is less for the smaller surface tension case. Also note that the wavelength of the interfacial waves along I_b appears smaller for smaller Σ_b. This is consistent with the scaling introduced in equation (2.4) and with a small-amplitude analysis, as in Keller & Miksis (1983), which shows that the surface waves should oscillate like the cube of the similarity variable. In addition, from equations (2.3) we see that in the limit $\Sigma_b \to 0$, then $\gamma_1 = \gamma_3 = \pi/2$ and $\gamma_2 = \pi$, i.e. the interfaces I_a and I_c appear to join smoothly and the problem appears to reduce to the motion of a single wedge with an initial wedge angle of 130°. This limit can be observed in figure 3.

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